

# The Shell Model, the Renormalization Group and the Two-Body Interaction

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The no-core shell model and the effective interaction  $V_{\text{low } k}$  can both be derived using the Lee-Suzuki projection operator formalism. The main difference between the two is the choice of basis states that define the model space. The effective interaction  $V_{\text{low } k}$  can also be derived using the renormalization group. That renormalization group derivation can be extended in a straight forward manner to also include the no-core shell model. In the nuclear matter limit the no-core shell model effective interaction in the two-body approximation reduces identically to  $V_{\text{low } k}$ . The same considerations apply to the Bloch-Horowitz version of the shell model and the renormalization group treatment of two-body scattering by Birse, McGovern and Richardson.

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*Ab initio* shell model calculations are becoming a staple of nuclear structure calculations[1, 2, 3, 4]. The motivation is to start with a bare two-nucleon potential or a bare two-nucleon plus bare three-nucleon potential and calculate nuclear properties from that potential making only controllable approximations. The potential is taken as given. A parallel development[5, 6, 7] is taking place in the study of the nucleon-nucleon potential where effective interactions are being developed through the use of renormalization group techniques. Here the motivation is not just to study nuclear structure but also to understand the nature of the two-body and many-body forces. It is the purpose of the present paper to show that these process should be considered convergent rather than parallel. The main and indeed the only significant difference between the two approaches is the choice of the basis that define the projection operator and hence the model space. The shell model uses the harmonic oscillator basis while the two-body work uses a plane wave basis.

To be more definite, first consider the no-core shell model. It uses the Lee-Suzuki projection formalism[8, 9]. A word is needed about what exactly is meant by the Lee-Suzuki formalism since there are variants that could be confusing if not kept straight. The key aspect[10] of the Lee-Suzuki method is the determination of the the op-

erator  $\omega$  defined by the equation:

$$Q\omega P|\psi\rangle = Q|\psi\rangle \quad (1)$$

where  $P$  and  $Q = 1 - P$  are the projection operators that define the model space. In the Lee-Suzuki formalism we define a set of states  $|k\rangle$  that are mapped into the model space defined by  $P$ . In terms of the states  $|k\rangle$  and a complete orthonormal set of states  $|\alpha_P\rangle$  that span the model space we have:

$$Q\omega P|\alpha_P\rangle = \sum_k Q|k\rangle\langle\alpha_P|k\rangle^{-1} \quad (2)$$

Once  $\omega$  is determined we can calculate[10] the effective interaction. We can have either a non-hermitian version of the interaction, which is simpler, or an hermitian version which is more complicated. For the present purposes the non-hermitian version is preferred.

In the no-core shell the projection operator,  $P$ , is defined in terms of harmonic oscillator states. For a given oscillator parameter,  $\hbar\Omega$ , the states up to a given energy are used. In addition an harmonic oscillator term in the center of mass is added. While this latter term is very important in the numerical work it plays no role in our discussion.

In the derivation of  $V_{\text{low } k}$ [6] the Lee-Suzuki formalism is also used but with the projection operator define by plane wave states. All plane wave states below a given momentum are included in the model space. Since  $V_{\text{low } k}$  can also be derived in a renormalization group approach it is normal to ask if, in general, the Lee-Suzuki formalism can

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be cast into a renormalization language. For a class of projection operators, including the one used in the no-core shell model, this is indeed possible. Once the formalism is set up it is trivial. The main difference with the derivation of  $V_{\text{low } k}$  is that we have discrete states.

The procedure is to recast the discrete state problem into a form that resembles the Lippmann-Schwinger equation. Then either  $V_{\text{low } k}$  or Birse formalism[5] can be applied line for line to get a flow equation for the shell-model interaction. We define a Hamiltonian  $H = H_o + \delta H$  and an unperturbed Hamiltonian  $H_o$  that is used to define the projection operator  $P = \theta(\Lambda - H_o) = \sum_n^N |\psi_o^n\rangle\langle\psi_o^n|$ . Here  $N$  denotes the highest energy state in the model space and the eigenvalue equations are given by:

$$H_o|\psi_o^n\rangle = E_o^n|\psi_o^n\rangle \quad (3)$$

$$H|\psi^n\rangle = E^n|\psi^n\rangle \quad (4)$$

Next a t-matrix like quantity is defined by:

$$T|\psi_o^n\rangle = \delta H|\psi^n\rangle \quad (5)$$

As in scattering theory (which this is not) we have the following equations:

$$|\psi^n\rangle = |\psi_o^n\rangle + G_o(E)\delta H|\psi^n\rangle \quad (6)$$

$$= |\psi_o^n\rangle + G_o(E)T|\psi_o^n\rangle \quad (7)$$

$$T = \delta H + \delta H G_o(E)T \quad (8)$$

Next we define the matrix elements  $T^{n',n} = \langle\psi_o^{n'}|T|\psi_o^n\rangle$  and  $V^{n',n} = \langle\psi_o^{n'}|\delta H|\psi_o^n\rangle$ . The symbol  $V$  is chosen to make the results look more familiar. The Lippmann-Schwinger equation is now written:

$$T^{n',n} = V^{n',n} + \sum_{n''} V^{n',n''} \frac{1}{E^n - E_o^{n''}} T^{n'',n} \quad (9)$$

The expansion coefficients defined by  $|\psi^n\rangle = \sum_{n'} a_{n'}^n |\psi_o^{n'}\rangle$  are given by:

$$a_{n'}^n = \delta^{n',n} + \frac{1}{E^n - E_o^{n'}} T^{n',n} \quad (10)$$

Note that there are no sums in this last equation.

The effect of the projection is to restrict the intermediate sums. We can now define an effective interaction by:

$$T^{n',n} = V_N^{n',n} + \sum_{n''}^N V_N^{n',n''} \frac{1}{E^n - E_o^{n''}} T^{n'',n} \quad (11)$$

where the effective interaction,  $V_N^{n',n}$ , is chosen to keep the half-off shell matrix  $T$  or the fully off shell  $T$  independent of the cutoff. It is here and in similar sums that the explicit form of the projection operator given above is needed.

We can now do two different developments. First we can follow the proponents of  $V_{\text{low } k}$  and work with the half-off shell version of the  $T$ -matrix. The procedure is exactly as they did[6] with the integrals replaced by sums. The resulting flow equation is:

$$V_N^{n',n}_{LS} - V_{N-1}^{n',n}_{LS} = V_N^{n',N}_{LS} \frac{1}{E_o^N - E_o^n} T^{N,n} \quad (12)$$

Note that it is the unperturbed energies that occur in this expression. By the arguments in ref. [10] this corresponds to the Lee-Suzuki projection formalism (hence the LS subscript). We stress again that it is the non-hermitian version (eq. 3 of ref.[10]) of the Lee-Suzuki interaction that agrees with the renormalization group interaction.

Second we can follow Birse *et al* [5] and use the fully off shell  $T$ -matrix to recover[10] the Bloch-Horowitz projection formalism. A less general derivation is given in ref. [11]. Again the only difference is that the integrals are replaced by sums. The resulting flow equation is:

$$V_N^{n',n}_{BH} - V_{N-1}^{n',n}_{BH} = V_N^{n',N}_{BH} \frac{1}{E_o^N - E} V_N^{N,n}_{BH} \quad (13)$$

where  $E$  is the value of the energy where the  $T$  matrix is required. This equation is valid for  $n \leq N-1$ . The value of  $\mathcal{N}$  is not uniquely determined but rather the equation is valid for any  $\mathcal{N} \geq N-1$ . For many purposes the value  $\mathcal{N} = N$  is the most useful.

The difference between eqs. 12 and 13 is the usual trade off between  $G_o T$  and  $GV$  and indeed the derivation of  $V_{\text{low } k}$  uses that equivalence. Eq. 12 requires that  $T$  be calculated at each step. The result is energy independent. On the other hand, eq. 13 just uses  $V$  but must be calculated separately for each energy. Although the two effective potentials are related[12] they have different properties[10] and are not equal. Since potentials are not observables, there is no particular reason they should be.

Since both the no-core shell model and the  $V_{\text{low } k}$  interaction can be derived from the renormalization group we see that using  $V_{\text{low } k}$  in a no-core shell-model calculation may not be very beneficial. Certainly there would be little gained in re-

renormalizing an interaction that has already been renormalized. Using  $V_{\text{low } k}$  directly in the oscillator diagonalization would probably be preferable as it already has suppressed the high momentum components.

The essential difference between shell models (either Lee-Suzuki or Bloch-Horowitz) and the corresponding continuum two-body calculations,  $V_{\text{low } K}$  or Birse *et al*[5], is the choice of projection operator: harmonic oscillator vs plane wave. However there is one limit in which the two become the same: namely the nuclear matter limit. This limit consists of letting the  $\hbar\Omega \rightarrow 0$ ,  $N \rightarrow \infty$  while keeping the cutoff energy,  $E_{\text{cutoff}}$  fixed (in one dimension  $(N + 1/2)\hbar\Omega = E_{\text{cutoff}}$ ). This is easiest to see in one dimension where the harmonic oscillator wave functions are a Gaussian times an Hermite polynomial. The asymptotic form for of the Hermite polynomial is given by (see ref. [13] eqs. 8.955) :

$$H_{2n}(x) = \frac{(-1)^n 2^{2n}}{\sqrt{\pi}} e^{x^2/2} \Gamma(n + 1/2) \cos(\sqrt{4n + 1}x) \quad (14)$$

with a similar equation for the odd Hermite polynomials in terms of sine functions. The argument of the cosine is just  $r\sqrt{2mE}/\hbar = rp/\hbar$  once the oscillator length parameter is inserted and  $n\hbar\Omega$  is expressed in terms of  $E$ . The coefficient of the cosine cancels the oscillator normalization to within a factor of  $\sqrt{\pi}/2$  when Sterling's formula (ref. [13] eq. 8.327) is used for the gamma functions. Since we are interested in large  $n$  this is valid.

Thus we see that in the nuclear matter limit the harmonic-oscillator shell-model projection operator reduces to the momentum space projection operator. If the shell model is carried out to the two-body cluster approximation then the effective interaction reduces identically to  $V_{\text{low } k}$  (Lee-Suzuki) or Birse *et al*[5] (Bloch-Horowitz). Therefore nuclear matter, studies such as ref. [14], are relevant to the no-core shell model even when  $V_{\text{low } k}$  is not explicitly used in the shell model calculation. Or even more interestingly refs. [1] and ref. [14] are partially discussing the same thing and coming to the same conclusion: namely that when done properly nuclear physics is perturbative. They did this although one was using Bloch-Horowitz in finite nuclei and the other was using Lee-Suzuki in nuclear matter.

Even more information relevant to the shell model and nuclear physics in general can be extracted from the nuclear matter results. Ref. [14],

working in nuclear matter, emphatically makes the point that a higher cutoff (momentum or energy) is not always better. Applying the same reasoning to finite nuclei we infer that shell-model calculations may not gain by taking larger and larger model spaces. But rather the model space should be matched to the energy and momentum scales of the problem since making the space larger may just induce spurious loop contributions that then have to be canceled, for example, by many-body forces. Ref. [15] (especially fig. 1) indicates that higher cutoffs require stronger many-body forces. Indeed nuclear physics has been plagued by the need to cancel loop contributions. The success of Dirac phenomenology[16] has been traced to the fact that it does a better job of suppressing[17, 18] high loop momenta than non-relativistic calculations. In pion-nucleus scattering there is similar[17] need to suppress[19, 20] high momentum loop contributions. In the later case this is frequently discussed under that name of the EELL effect[21].

Interactions with low cutoffs, in either momentum or oscillator space, cannot be considered more or less fundamental than those with high cutoffs. This is especially true for those obtained in the Lee-Suzuki formalism since a unitary transformation underlies[9] this approach. Unitary transformations change the appearances but not the predictions or underlying physics. Thus one can not say  $V_{\text{low } k}$  is any more or less fundamental than the higher cutoff potentials it is related to. The arguments apply equally well to the shell model effective interactions. Does the use of an harmonic oscillator basis make them less fundamental than those in a plane wave basis? Thus we can consider the no-core shell-model model-space interactions just as fundamental as  $V_{\text{low } k}$ . In nuclear matter they become the same anyway. This discussion is not meant as a criticism of  $V_{\text{low } k}$  but rather meant to suggest a different way of looking at both  $V_{\text{low } k}$  and the shell model effective interaction.

We noted at the beginning the rather different motivations for the shell model and two-body continuum calculations: the difference between using and generating an interaction. The discussion in this paper suggests the distinction is not all that clear cut. The interaction must be matched to the model space. Even the concept of a fundamental interaction is fraught with difficulties.

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